ON THE CONSTRUCTION OF CERTAIN 6-DIMENSIONAL SYMPLECTIC MANIFOLDS WITH HAMILTONIAN CIRCLE ACTIONS

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ABSTRACT. Let (M, ω) be a connected, compact 6-dimensional symplectic manifold equipped with a semi-free Hamiltonian S^1 action such that the fixed point set consists of isolated points or surfaces. Assume dim $H^2(M) < 3$, in [L], we defined a certain invariant of such spaces which consists of fixed point data and twist type, and we divided the possible values of these invariants into six "types". In this paper, we construct such manifolds with these "types". As a consequence, we have a precise list of the values of these invariants.

1. INTRODUCTION

Assume (M, ω) is a connected, compact 6-dimensional symplectic manifold equipped with a semi-free (free outside fixed point set) Hamiltonian S^1 action such that the fixed point set consists of isolated points or surfaces. For the case dim $H^2(M) < 3$, in [L], we defined a certain invariant which consists of the "fixed point data" and "twist type", and we gave a complete list of the possible values of these invariants. We have seen that only six "types" can occur (see Theorem 1 below). We showed the existences for a few cases by giving explicit examples, such as coadjoint orbits or toric varieties. In this paper, we illustrate a method for constructing such manifolds. As a consequence, this proves the existences of the manifolds with those prescribed types as in [L]. The techniques we use in this paper are quite different from those we used in [L]. We mainly use symplectic topological method and related results on symplectic 4-manifolds.

Recall that the moment map ϕ is a perfect Morse-Bott function. The fixed point sets of the circle action are exactly the critical sets of ϕ. Each critical set has even index. ϕ has a unique local minimum and a unique local maximum ([A]). Recall the following definitions from [L]:

Definition 1. Let (M, ω) be a connected, compact 6-dimensional symplectic manifold equipped with a semi-free Hamiltonian S^1 action such that the fixed point set consists of isolated points or surfaces. Assume dim $H^2(M) < 3$. For each such manifold, we define the **fixed point data** to be the diffeomorphism type of the fixed point sets, their indices, and b_{min}, where b_{min} denotes the 1st Chern number of the normal bundle of the minimum when the minimum is a surface.

If the minimum (or the maximum) is a surface, then for a regular value right above the minimum (or right below the maximum), the reduced space is diffeomorphic to the total space of an S^2 fibration over the minimum (or over the maximum).

When the fixed point set consists only of surfaces, the reduced spaces at the non-extremal values are all diffeomorphic, and there is a canonical isomorphism between their (co)homology groups (see [GS]).

Definition 2. Let (M, ω) be a compact connected 6-dimensional symplectic manifold equipped with a semi-free Hamiltonian circle action. Assume the fixed point set consists only of surfaces. If the fibers of the reduced spaces at a regular level right above the minimum and right below the maximum can be represented by the same cohomology class of the reduced space, we say there is **no twist**; otherwise, we say there is a **twist**.

In [L], we have shown the following

Theorem 1. *Let (M, ω) be a connected compact 6 dimensional symplectic manifold equipped with a semi-free Hamiltonian circle action such that the fixed point set consists of isolated points or surfaces. Assume* $\dim H^2(M) < 3$. *Then the possible fixed point data and the twist type are:*
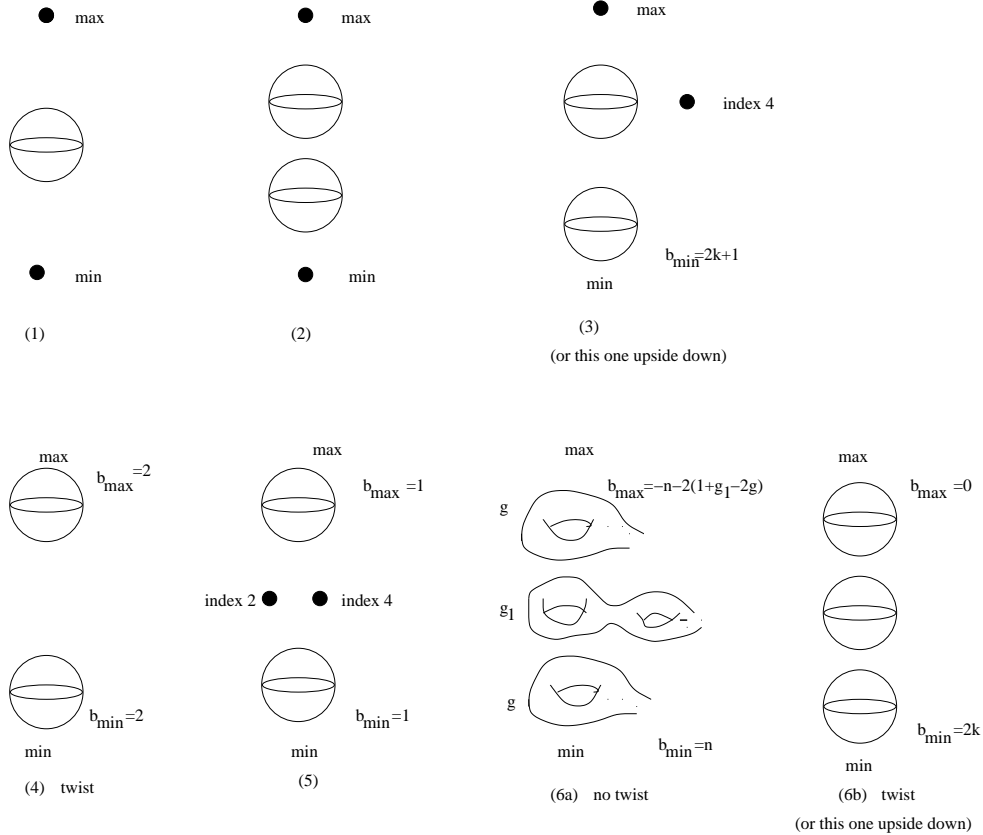
In the picture, g, g_1 are the genuses of the surfaces, and b_{max} is the 1st Chern number of the normal bundle of the maximum.

Remark 1. The picture gives us an order of how the moment map crosses the fixed point sets. For instance, manifold of type (1) means that the manifold has an isolated minimum, an index 2 sphere, and an isolated maximum.

For manifold of type (2), the two index 2 spheres can not be on the same level of the moment map.

For manifolds of type (3) and (5), the moment map can cross the two non-extremal fixed point sets in any order or the 2 fixed point sets are on the same level of the moment map.

Remark 2. In [L], we have seen that when there is an index 2 surface or the maximum is a surface in a certain "type", the 1st Chern numbers of the normal bundles of these surfaces are uniquely determined by the fixed point data and the twist type. (This is the reason that b_{max} is not included in the definition of the **fixed point data**.) For instance, for manifolds of type (1), the 1st Chern numbers of the positive normal bundle and the negative normal bundle of the index 2 sphere are both 2; for manifolds of type (2), the 1st Chern numbers of the positive normal bundle and the negative normal bundle of the 1st (counting from bottom) index 2 sphere are 0 and 1 respectively, and those of the second index 2 sphere are respectively 1 and 0. Notice that in type (4) and type (5), b_{min} and b_{max} have fixed values, in types (3), (6a) and (6b), on the other hand, b_{min} can vary, and in particular, for type (6a), the genuses g and g_1 can vary. Different values of these "variables" of course give different manifolds.

Remark 3. We have shown in [L] that there exists a manifold of type (1) which is a coadjoint orbit of $SO(5)$. We constructed toric varieties for type (3) when $b_{min} = 1$ and $b_{min} = 3$, for type (4) and for type (6b) with $b_{min} = 2$. Moreover, we have shown that all manifolds of type (4) are diffeomorphic to $\mathbb{C}P^3$, and all manifolds of type (6b) with $b_{min} = 2$ are diffeomorphic to the corresponding toric variety which is $\mathbb{C}P^3$ blow up at a point.

In [L] and in this paper, the **fixed point data** has a certain meaning, for the sake of convenience. More generally, similar data (looser or stronger) is an important tool for the studying of symplectic Lie group actions, especially for the studying of Hamiltonian Lie group actions. It captures important geometrical and topological information about the manifolds. As we have seen in [L], for a fixed "type" in Theorem 1, the **fixed point data** uniquely determines the equivariant (ordinary) cohomology ring and equivariant (ordinary) Chern classes of the manifold. More significantly, for some cases, this data classifies the manifolds up to diffeomorphism (see [L]). Recall that for all the types of manifolds in Theorem 1, the reduced spaces (except at the extremal values) are $\mathbb{C}P^2$ or **ruled manifolds**. We understand well the symplectic structures of these 4-manifolds and their symplectic submanifolds. In this paper, we use this special feature to construct manifolds with the types prescribed in Theorem 1. We will show that for each "type" in Theorem 1, except for type (6a) with $1 + g_1 - 2g \leq 0$, this "local" data (related in a global way, however) allows us to construct a "global" manifold with the prescribed data. In fact, except for type (6a) with $1 + g_1 - 2g \leq 0$, this data uniquely determines the inverse image (under the moment map) of a small neighborhood of each critical or regular value of the moment map up to equivariant symplectomorphism (see Section 2). Let us call these "local pieces". A "local piece" for the minimum or the maximum can always be constructed. The main problem is to construct a "local piece" for an index 2 surface. In this case, we need 4 pieces of information (M_0, ω_0, P, X) (see Section 2 for its meaning), where M_0 is the reduced space at this critical level, and X has to be a symplectic submanifold of M_0. More precisely,

X is the symplectically embedded image of the index 2 surface in M_0 (see Section 2). The prescribed type must make this to be true. Of course, the prescribed type must guarantee that each reduced space is symplectic, i.e., the cohomology classes of the reduced symplectic forms (we can only know the cohomology classes of the forms) do contain symplectic forms (see Section 3). For (6a), if $1 + g_1 - 2g \leq 0$, the dual class (which can be computed from the "type") of the surface in the reduced space can not represent the symplectically embedded image of the index 2 surface. That is the reason we failed to construct such manifolds for type (6a) when $1 + g_1 - 2g \leq 0$. In order to construct the manifolds globally, we need to perform some gluing along some regular levels. The main reason we can perform the gluing is that for a regular level of the moment map, two different symplectic forms in the same cohomology class of the reduced space (which is $\mathbb{C}P^2$ or a **ruled manifold**) are diffeomorphic. These classification results on the symplectic structures on $\mathbb{C}P^2$ or **ruled manifolds** are due to Taubes and Lalonde-McDuff ([T], [LMc]).

In [L] and in this paper, we assumed that dim $H^2(M) < 3$. Under this assumption, we defined **fixed point data**. Notice that the definition of **twist** does not depend on this restriction, it is defined when the fixed point set consists only of surfaces. As we mentioned in Remark 2, when there is an index 2 surface or the maximum is a surface in a certain "type", the 1st Chern numbers of the normal bundles of these surfaces are uniquely determined by the fixed point data. The "twist type" controls the "top" and the "bottom". If we remove the assumption dim $H^2(M) < 3$, we have more fixed point set components. If the fixed point set consists only of surfaces, we still have our definition of **twist**. If the fixed point set also consists index 2 or index 4 isolated points, then the reduced spaces change not only by diffeomorphism, but also by blow up or blow downs ([GS]). The reduced spaces and their cohomologies are more complicated. The definition of **twist** needs to be properly addressed. For the definition of **fixed point data**, we need to specify more 1st Chern numbers of the normal bundles of the fixed point set components to determine those of the other fixed point set components. Alternatively, depending on the purposes, we may assume that all these Chern numbers for all the fixed point set components are compatibly given and we include all of them in the definition. For the construction of these more complicated manifolds, we may not understand the symplectic structures on the reduced spaces well enough to do the local construction or to perform the global gluing. Certainly, in good cases, if our requrements are satisfied, we can still construct the manifolds with certain "types" other than those in Theorem 1.

In summary, in order to construct a global manifold from a prescribed type, we need the "data" to be "compatibly matched", and we need the reduced spaces to be "good enough" to allow us to perform the gluing. The idea itself can be applied to more general constructions of Hamiltonian S^1 manifolds, for instance, for non-semi-free Hamiltonian circle actions on other dimensional symplectic manifolds with other fixed point set components. Or more generally, depending on what is given, the idea itself may be applied to the setting with no Hamiltonian circle actions.

The main result of this paper is

Theorem 2. *There exists a manifold for any of the prescribed fixed point data and twist type as in Theorem 1, except that for type (6a), the genuses g and g_1 must satisfy $1 + g_1 - 2g > 0$.*

Recall that for type (6a), we computed the dual class η of the index 2 surface in the reduced space at the critical level in [L]: $\eta = (1 + g_1 - 2g)x + 2y$ for even n, and $\eta = (2 + g_1 - 2g)x + 2y$ for odd n (see Section 3 for the meaning of x and y). If $1 + g_1 - 2g \leq 0$, then these dual classes can not represent the connected symplectically embedded index 2 surface in the reduced spaces. The class $2y$ (for even n) or $x + 2y$ (for odd n) represents non-connected symplectic submanifolds. The classes $\eta = (1 + g_1 - 2g)x + 2y$ or $\eta = (2 + g_1 - 2g)x + 2y$ with $1 + g_1 - 2g < 0$ can not represent the symplectically embedded image of the genus g_1 index 2 surface due to the following reason: the projection of the embedded surface (represented by $\eta = (1 + g_1 - 2g)x + 2y$ or $\eta = (2 + g_1 - 2g)x + 2y$) to the "base" of the S^2 fibration (represented by y) is a smooth degree 2 map from a genus g_1 Riemann surface to a genus g Riemann surface. By the analogue Hurwitz formula for smooth maps between Riemann surfaces (see Theorem 1 in [HY]), g_1 and g must satisfy $1 + g_1 - 2g \geq 0$. This is the reason we can not construct such manifolds. This fact also helps us to strengthen Theorem 1: [1]

Corollary 1. For type (6a) in Theorem 1, the genuses of the surfaces must satisfy $1 + g_1 - 2g > 0$.

Theorem 1 and this corollary give us a complete list of the possible "types". Theorem 2 shows that there exist manifolds of all these "types". Together, they give us exactly the "types" of manifolds that can occur under the assumptions of Theorem 1.

Acknowledgement. I would like to thank Denis Auroux for explaining to me the symplectic submanifolds of ruled surfaces. I thank him for explaining to me the Hurwitz formula for complex maps between Riemann surfaces. For type (6a), the idea of the proof of the non-existence of the symplectic submanifolds in ruled surfaces for the cases $1 + g_1 - 2g \leq 0$ is due to him.

I would like to thank the referee for his invaluable comments and constructive suggestions.

2. Construction of neighborhoods of critical and regular level sets of the moment map

Construction of a neighborhood of the minimum:

If the minimum is isolated, by the equivariant Darboux-Weinstein theorem, a neighborhood of the minimum is equivariantly symplectomorphic to $\mathbb{C}^3$ with the standard symplectic form and S^1 action with weights $1, 1, 1$.

If the minimum is a surface Σ_g, then a neighborhood of the minimum is a $\mathbb{C}^2$ bundle E over Σ_g. It is easy to construct such a bundle with 1st Chern number b_{min}. (We can always construct 2 line bundles with Euler numbers m and n for any integer m and n and take the sum of the line bundles.). We can put a symplectic form on a neighborhood of the zero section of E (see [GS]) and make S^1 act semi-freely by fixing the base and rotating the fibers. More precisely, let F be the unitary frame bundle of E (assume there is a Hermitian metric on E), T^*F be the cotangent bundle and ω_F be the canonical symplectic form on T^*F. A connection on F can give us the vertical cotangent bundle V^*F. Let $i : V^*F \hookrightarrow T^*F$ be the inclusion,

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[1] The construction procedure gave me a good reason to study the symplectic submanifolds of the reduced spaces. This helped me to discover this piece of information.

and let ω_B be the symplectic form on the base Σ_g. Then $\omega_V = i^*\omega_F + \pi^*\omega_B$ is a symplectic form on V^*F. The $U(2)$ action on V^*F is Hamiltonian, so is the $U(2)$ action on $\mathbb{C}^2$. Take $V^*F^- \times \mathbb{C}^2$ with the product symplectic structure, where V^*F^- is V^*F with the form $-\omega_V$. There is a Hamiltonian S^1 action on this space: it acts trivially on the first factor and it acts on the second factor by rotation with weights $1, 1$. We can get a neighborhood of the zero section of E with a symplectic form by reducing the $U(2)$ action at 0. The semi-free Hamiltonian S^1 action remains on E. By the equivariant Darboux-Weinstein theorem, any S^1-invariant symplectic structure on the above neighborhood is equivariantly isomorphic to the above one.

Notice that the above construction works for other dimensions and for non-semi-free S^1 actions.

Hence, we have the following lemma for the construction of a neighborhood of the minimum.

Lemma 1. *Assume F is an isolated point or is a closed surface, then we can construct uniquely up to isomorphism a(n) (open) Hamiltonian S^1 symplectic manifold $M_{[min,\epsilon_0)}$ such that F is the minimum of the moment map, and such that F has any prescribed 1st Chern number of its normal bundle in $M_{[min,\epsilon_0)}$ when F is a surface.*

To construct a neighborhood of a critical level which contains an index 2 point or an index 2 surface, we can use the result of Guillemin and Sternberg ([GS]):

Assume we have (M_0, ω_0, P, X), where M_0 is a compact symplectic manifold with symplectic form ω_0, X is a symplectic submanifold of M_0, and P is a principal S^1 bundle over M_0, $\pi : P \to M_0$. Given a small open interval $I = (-\epsilon, +\epsilon)$, we can construct a symplectic manifold M_I with a semi-free Hamiltonian S^1 action with proper moment map ϕ which maps M_I onto I. The symplectic form on M_I is determined by ω_0, the normal bundle of X in M_0 and the Euler class of the circle bundle P over M_0. Moreover, 0 is an index 2 critical level of ϕ. For $-\tau < 0$, $\phi^{-1}(-\tau)$ is diffeomorphic as an S^1 space to P, the reduced spaces of M_I at $-\tau$ and 0 are diffeomorphic to M_0, and the reduced space of M_I at $\tau > 0$ is the blow up of M_0 along X. The reduced symplectic forms satisfy the Duistermaat-Heckman formula ([DH]). Moreover, M_I is unique up to S^1- equivariant symplectomorphisms (Theorem 13.1 in [GS]).

We summarize this in

Lemma 2. *Assume we have (M_0, ω_0, P, X), where M_0 is a compact symplectic manifold with symplectic form ω_0, X is a symplectic submanifold of M_0, and P is a principal S^1 bundle over M_0, $\pi : P \to M_0$. Given a small open interval $I = (-\epsilon, +\epsilon)$, we can construct uniquely up to isomorphism a semi-free Hamiltonian S^1 symplectic manifold M_I with moment map ϕ (which maps M_I onto I) such that $\phi^{-1}(0)$ is a critical level with an index 2 fixed point set which is diffeomorphic to X, $\phi^{-1}(-\tau)$ is diffeomorphic as an S^1 space to P, and the reduced spaces of M_I at $-\tau$ and 0 are diffeomorphic to M_0, the reduced space of M_I at $\tau > 0$ is the blow up of M_0 along X.*

We can similarly construct a neighborhood of a critical level set which contains an index 4 fixed point set component by looking at it upside down.

The construction of a neighborhood of the maximum is similar to the construction of a neighborhood of the minimum.

The above construction of neighborhoods of critical levels can only be done for small intervals of critical values of ϕ. When we construct our manifolds, we may need a "longer" neighborhood of a regular level. For this purpose, we construct a "regular neighborhood".

If I is a connected open interval consisting of regular values of ϕ, then $\phi^{-1}(I)$ is S^1-equivariantly diffeomorphic to $\phi^{-1}(a) \times I$ where $a \in I$. The S^1-invariant symplectic form on $\phi^{-1}(I)$ is determined by the symplectic forms on the reduced spaces and the Euler class of the principal circle bundle $\phi^{-1}(a)$ over M_a.

Lemma 3. ([Mc1]) *For an interval I consisting of regular values of ϕ, if ω is the symplectic form on $\phi^{-1}(I)$, and τ_s is the reduced symplectic form at $s \in I$, then ω is determined by the forms τ_s up to an S^1 equivariant diffeomorphism which preserves the level sets of ϕ.*

See [Mc1] for details.

Assume that these neighborhoods we constructed above are "big enough" so that we can shrink them a little such that they are manifolds with boundaries, the boundaries being regular level sets of ϕ. We will call these neighborhoods with boundaries of critical levels and regular levels "local pieces".

3. Problems for local and global constructions

We have seen in Section 2 what we need to prescribe to do the local construction. Let us review in this section what information the "types" in Theorem 1 can give us and what we need to take care of when we do the local and global constructions.

Assume a is a regular value of the moment map ϕ, then S^1 acts freely on $\phi^{-1}(a)$, so we have a principal circle bundle

$$(1) \qquad \begin{array}{ccc} S^1 & \hookrightarrow & \phi^{-1}(a) = P_a \\ & & \downarrow \\ & & \phi^{-1}(a)/S^1 = M_a, \end{array}$$

where M_a is the reduced space at a.

Assume the minimum (or the maximum) of the moment map is a surface Σ_g. Let a be a regular value right above the minimal value (or right below the maximal value) such that there are no other critical sets in the neighborhood except the minimum (or the maximum). Then M_a is the total space of a sphere bundle over Σ_g. Let $S^2 \times \Sigma_g$ be the trivial bundle, and let E_{Σ_g} be the non-trivial bundle. Let $e(P_a)$ be the Euler class of the circle bundle P_a over M_a.

For $S^2 \times \Sigma_g$, let $x, y \in H^2(S^2 \times \Sigma_g)$ be a basis which consists of the dual classes of the fiber S^2 and the base Σ_g respectively. Then $\int_{base} x = \int_{S^2 \times \Sigma_g} xy = 1$, $\int_{fiber} y = \int_{S^2 \times \Sigma_g} yx = 1$, $\int_{fiber} x = \int_{S^2 \times \Sigma_g} x^2 = 0$, and $\int_{base} y = \int_{S^2 \times \Sigma_g} y^2 = 0$.

Similarly, for the non-trivial bundle E_{Σ_g}, let $x, y \in H^2(E_{\Sigma_g})$ be a basis which consists of the dual classes of the fiber and the section Σ_- which has self-intersection -1 respectively. Then $\int_{\Sigma_-} x = \int_{E_{\Sigma_g}} xy = 1$, $\int_{fiber} y = \int_{E_{\Sigma_g}} xy = 1$, $\int_{fiber} x = \int_{E_{\Sigma_g}} x^2 = 0$, and $\int_{\Sigma_-} y = \int_{E_{\Sigma_g}} y^2 = -1$.

We have the following lemmas

Lemma 4. *([L]) If the minimum is a surface Σ_g with 1st Chern number b_{min} of its normal bundle, then M_a (a above the minimum) is diffeomorphic to $S^2 \times \Sigma_g$ if and only if $b_{min} = 2k$ is even, and it is diffeomorphic to E_{Σ_g} if and only if $b_{min} = 2k+1$ is odd. In either case, $e(P_a) = kx - y$.*

Lemma 5. *([L]) If the maximum is a surface Σ_g with 1st Chern number b_{max} of its normal bundle, then M_a (a below the maximum) is diffeomorphic to $S^2 \times \Sigma_g$ if and only if $b_{max} = 2k'$ is even, and it is diffeomorphic to E_{Σ_g} if and only if $b_{max} = 2k' + 1$ is odd. In either case, $e(P_a) = -k'x + y$.*

Let c be a critical value of the moment map ϕ, and $S \subset \phi^{-1}(c)$ be an index 2 fixed surface. Assume for small ϵ, S is the only critical set in $\phi^{-1}([c-\epsilon, c+\epsilon])$. Let $P_{c-\epsilon}$ be the circle bundle $\phi^{-1}(c - \epsilon)$ over $M_{c-\epsilon}$, $e(P_{c-\epsilon})$ be its Euler class. $P_{c+\epsilon}$ and $e(P_{c+\epsilon})$ have similar meanings. By [GS] or [Mc1], $M_{c+\epsilon}$ is diffeomorphic to $M_{c-\epsilon}$ (in dimension 6), let's call them M_{red} (reduced space). Moreover, there is a symplectic embedding $i : S \to M_{red}$ such that if $i(S) = Z$, and η is the dual class of Z in M_{red}, then

$$(2) \qquad\qquad e(P_{c+\epsilon}) = e(P_{c-\epsilon}) + \eta.$$

By [GS], when the moment map crosses an index 2 point (index 4 point), the reduced space changes by a blow up at a point (blow down of an exceptional divisor). Now let M_- be the reduced space before surgery (blow up or blow down), M_+ be the reduced space after surgery, and let $e(P_-)$ be the Euler class of the principal circle bundle over M_-, $e(P_+)$ be the Euler class of the principal circle bundle over M_+. Let $\eta_{ex,div}$ be the dual class of the exceptional sphere in M_+ after blow up or in M_- before blow down. Let's always call the blow down map by β. By [GS], we have the following:

Corresponding to blow up,

$$(3) \qquad\qquad e(P_+) = \beta^* e(P_-) + \eta_{ex,div}.$$

Corresponding to blow down,

$$(4) \qquad\qquad \beta^* e(P_+) = e(P_-) + \eta_{ex,div}.$$

For each type of manifolds in Theorem 1, we can use Lemma 4, Lemma 5 and equations (2), (3), (4) to compute the Euler classes of the principal S^1 bundles (1) for regular levels in terms of the fixed point data. By Duistermaat-Heckman formula ([DH]), we can compute the cohomology classes of the reduced symplectic forms on the reduced spaces.

Recall that for the types of manifolds in Theorem 1, the reduced spaces are all smooth manifolds (see Section 10 in [GS]). Moreover, from above we see that (except at the minimal value and at the maximal value) they are diffeomorphic to either $\mathbb{C}P^2$ or the total space of an S^2 bundle over a Riemann surface (assume for manifold of type (5) the moment map crosses the index 4 point before it crosses the index 2 point). Now, let us list the problems we should take care of for the construction:

1. The cohomology classes of the reduced forms given by Duistermaat-Heckman formula are in terms of the fixed point data, the size of the minimum (maximum)

if the minimum (maximum) is a surface, and the distances between the moment map images of the fixed point set components. The question is which cohomology class in the reduced space contains a symplectic form.

2. If F is an index 2 fixed surface, then F is symplectically embedded in the reduced space at the critical level (and nearby regular levels). As we have seen (see [L] for details), in each case, the dual cohomology class of F in the reduced space can be computed in terms of the fixed point data. The question is which dual class can be represented by a symplectic submanifold.

3. When we glue the "local pieces" together, we glue them along a regular level. Since this regular level is a principal S^1 bundle over the reduced space, we can glue the two reduced spaces on the local pieces together (the fibers will be identified correspondingly). We only know the cohomology classes of the reduced symplectic form are the same on these two reduced spaces, the symplectic forms may be chosen differently. The question is if there is a diffeomorphism between the two reduced spaces such that one form is pulled to another by this diffeomorphism.

Section 4 and Section 5 will give answers to the above questions.

4. Symplectic $\mathbb{C}P^2$, symplectic ruled manifolds, and their symplectic submanifolds

Lemma 6. *Let $f : (M, \omega_1) \to (M, \omega_2)$ be a diffeomorphism such that $f^* \omega_2 = \omega_1$. If $A \subset (M, \omega_1)$ is a symplectic submanifold, then $f(A) \subset (M, \omega_2)$ is a symplectic submanifold.*

Proof. $f_*(TA) = T(f(A))$ is an isomorphism. Take $X, Y \in TA$, then $\omega_2(f_* X, f_* Y) = f^* \omega_2(X, Y) = \omega_1(X, Y)$. $\qquad\square$

$\mathbb{C}P^2$ with its standard complex structure carries Kähler forms. Each complex submanifold of $\mathbb{C}P^2$ is symplectic with respect to these Kähler forms.

For the symplectic structures on $\mathbb{C}P^2$, we have

Theorem 3. *([T]) If (M, ω) is a symplectic manifold and M is smoothly diffeomorphic to $\mathbb{C}P^2$, then ω is diffeomorphic to a standard Kähler form.*

Lemma 7. *Let $u \in H^2(\mathbb{C}P^2)$ be a generator. Then a cohomology class $a = lu$ contains a symplectic form of $\mathbb{C}P^2$ if and only if $l \neq 0$.*

Proof. Any non-zero multiple of the standard symplectic form on $\mathbb{C}P^2$ is symplectic. $\qquad\square$

Lemma 8. *For $(\mathbb{C}P^2, \omega)$, let $u \in H^2(\mathbb{C}P^2, \mathbb{Z})$ be a generator. Then any class $\eta = au$ with a being a positive integer can be represented by a connected symplectic submanifold.*

Proof. By Theorem 3, ω is diffeomorphic to a stardard Kähler form. Since $\eta = au$ with a being a positive integer represents a connected complex curve for the standard complex structure, it represents a connected symplectic submanifold for the standard Kähler form. The lemma follows from Lemma 6. $\qquad\square$

Definition 3. An S^2 fibration over a Riemann surface $\pi : E \to \Sigma$ is called a **ruled manifold**.

Ruled manifolds are symplectic manifolds. One can choose the fibration such that the symplectic forms are nondegenerate on the fibers.

Definition 4. A symplectic form on a ruled manifold is **compatible** with the ruling $\pi : E \to \Sigma$ if it is nondegenerate on the fibers.

The following fact is well known

Lemma 9. *Let E be a ruled manifold, and F be the fiber of E. Any cohomology class $a \in H^2(E)$ which satisfies the following conditions carries a symplectic form which is nondegenerate on the fibers.*
(a) $a(F)$ and $a^2(E)$ are positive, and
(b) $a^2(E) > (a(F))^2$ if the bundle is non-trivial.

With this symplectic form and a compatible complex structure, every connected complex submanifold is a connected symplectic submanifold.

By D. McDuff, we have

Lemma 10. *([Mc2]) Let E be a ruled manifold, and F be the fiber of E. Then the cohomology class $a \in H^2(E)$ of any symplectic form on E which is compatible with the ruling satisfies the conditions*
(a) $a(F)$ and $a^2(E)$ are positive, and
(b) $a^2(E) > (a(F))^2$ if the bundle is non-trivial.

For the symplectic forms in the same cohomology class on a ruled manifold, we have

Theorem 4. *([LMc]) Let ω_0, ω_1 be two cohomologous symplectic forms on the ruled 4-manifold $\pi : E \to \Sigma$, then there is a diffeomorphism Φ of E such that $\Phi^*(\omega_0) = \omega_1$. Moreover, if we assume that the forms ω_0, ω_1 are both compatible with π, then they are isotopic.*

We have the following lemmas about the connected symplectic submanifolds of ruled surfaces.

Lemma 11. *For $(\Sigma_g \times S^2, \omega)$, where ω is compatible with the ruling, let x, y be the dual class of the fiber S^2 and the dual class of the base Σ_g respectively. Then x, y, and any class $ax + by$ with a, b being positive integers can be represented by a connected symplectic submanifold.*
If $\int [\omega] \cdot y > n \int [\omega] \cdot x$ ($n = 1, 2, ...$), then $y - nx$ can be represented by a connected symplectic submanifold.

Lemma 12. *For the non-trivial S^2 bundle (E_{Σ_g}, ω) over Σ_g, where ω is compatible with the ruling, let x, y be the dual class of the fiber S^2 and the dual class of the section with self-intersection -1 respectively. Then x, y, $x+y$, and any class $ax+by$ with integers $a > 1, b > 0$ can be represented by a connected symplectic submanifold.*
If $\int [\omega] \cdot y > n \int [\omega] \cdot x$ ($n = 1, 2, ...$), then $y - nx$ can be represented by a connected symplectic submanifold.

Proof of Lemma 11:

Proof. The classes x and y clearly represent symplectic submanifolds. For the classes $ax + by$ with a, b being positive integers, we can construct their symplectic representatives. Take the class $p_i \times S^2$, where $i = 1, 2, ..., a$, and take the class $\Sigma_g \times q_j$, where $j = 1, 2, ..., b$. These are symplectic submanifolds and they intersect positively transversely at ab points. Then one can smooth out the intersection points to obtain a smooth submanifold which is symplectic and represented by the class $ax + by$.

When the symplectic area of the base is n times larger than that of the fiber, the class $y - nx$ represents the section at ∞ of the projectivised bundle $P(L \oplus \mathbb{C})$, where L is a complex line bundle of degree $2n$ over Σ_g. So it represents a symplectic submanifold. $\qquad\square$

Alternatively, one may choose compatible complex structures on ruled manifolds and argue that some classes represent complex submanifolds. The proof of Lemma 12 is similar.

5. Gluing neighborhoods together

In Section 2, we constructed "local pieces". We will glue these "local pieces" together along their boundaries. Let L_1 and L_2 be two "adjacent" "local pieces" such that $\phi|L_1 \leq \phi|L_2$. Let a be the regular value such that $\phi^{-1}(a)$ is the level along which we glue L_1 and L_2. Suppose $\phi_-^{-1}(a)$ is the boundary of L_1, $\phi_+^{-1}(a)$ is the boundary of L_2, and we will identify them to get $\phi^{-1}(a)$. Now let $M_a^- = \phi_-^{-1}(a)/S^1$, and $M_a^+ = \phi_+^{-1}(a)/S^1$. Then $\phi_-^{-1}(a)$ is a principal circle bundle over M_a^-, and $\phi_+^{-1}(a)$ is a principal circle bundle over M_a^+. Let ω_- and ω_+ be the reduced symplectic forms on M_a^- and on M_a^+ respectively. We state the following lemma for convenience, it is obviously valid.

Lemma 13. *Assume that the two principal circle bundles $\phi_-^{-1}(a) \to M_a^-$ and $\phi_+^{-1}(a) \to M_a^+$ have the same Euler class. If there is a symplectomorphism g between (M_a^-, ω_-) and (M_a^+, ω_+), then g induces an isomorphism between $\phi_-^{-1}(a)$ and $\phi_+^{-1}(a)$.*

In our case, of course the two circle bundles have the same Euler class. As we have seen in Section 3, our reduced spaces are diffeomorphic to $\mathbb{C}P^2$ or diffeomorphic to a ruled manifold. M_a^- and M_a^+ are diffeomorphic as smooth manifolds. The cohomology classes of the reduced symplectic forms ω_- on M_a^- and ω_+ on M_a^+ agree. By Theorem 3 or by Theorem 4, there is a diffeomorphism $g : (M_a^-, \omega_-) \to (M_a^+, \omega_+)$ such that $g^* \omega_+ = \omega_-$. By Lemma 13, this symplectomorphism g between the reduced spaces induces an isomorphism between $\phi_-^{-1}(a)$ and $\phi_+^{-1}(a)$. We call the identified level set $\phi^{-1}(a)$.

6. Proof of Theorem 2

We are now ready to construct our manifolds. It will boil down to the question of choosing some parameters suitably.

Proof. In the following, we always use α_0 to denote the size of the minimum if the minimum is a surface, and use t_0, t_1, t_2 to denote the distances between the moment map images of the fixed point set components as the moment map goes from the minimum to the maximum. For convenience, we will use the following pictures. On the left, between the critical sets, we indicate the Euler class of the principal S^1 bundle over the reduced space at that regular interval, and on the right of the index 2 surface, we indicate the dual class of the index 2 surface in the reduced space.

By Section 5, the gluing of the "local pieces" can always be done, therefore we won't "glue explicitly" in the following proof, we will check requirements **1**. and **2**. listed in Section 3.

For manifold of type (1), we have seen an example in [L]. Let us use the construction method to prove its existence. We have the following picture ($u \in H^2(\mathbb{C}P^2)$ is a generator):
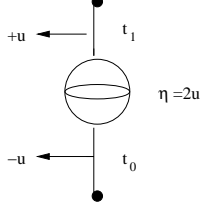
The reduced space at any non-extremal level is $\mathbb{C}P^2$.

Since the minimum and the maximum are unique, $t_0 > 0$, and $t_1 > 0$.

Let us use Duistermaat-Heckman formula ([DH]) to compute the cohomology classes of the reduced symplectic forms. If $0 < t \leq t_0$, then $[\omega_t] = ut$, by Lemma 7, for each t in this interval, $[\omega_t]$ contains a symplectic form.

By Lemma 8, the index 2 sphere is a symplectic submanifold of $\mathbb{C}P^2$.

Starting new at t_0, by Duistermaat-Heckman formula, for $0 < t \leq t_1$, $[\omega_t] = t_0 u - ut$. Since $[\omega_{t_1}] = 0$, $t_1 = t_0$. By lemma 7 again, each $[\omega_t]$ in this interval contains a symplectic form.

This proves the existence of manifold of type (1) in Theorem 1.

We can similarly prove the existence of manifold of type (2) in Theorem 1.

For manifold(s) of type (3), we saw in [L] that only when $b_{min} = 1$, ϕ can cross the index 4 point before it crosses the index 2 sphere. In this case, we constructed a toric variety. So let us assume the moment map crosses the critical sets in the following order.
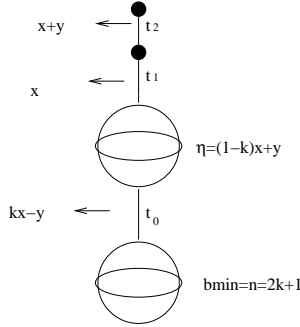
Since the minimum and the maximum are unique, $t_0 > 0, t_2 > 0$.

Note that up to the index 4 critical level (excluding this critical level and the minimal level), the reduced space is $\mathbb{C}\tilde{P}^2$ ($\mathbb{C}P^2$ blown up at a point). By Duistermaat-Heckman formula, if $0 < t \leq t_0$, then $[\omega_t] = \alpha_0 x - (kx - y)t = (\alpha_0 - kt)x + ty$. To make $[\omega_t]$ contain symplectic forms for each t, by Lemma 9, we can choose $\alpha_0 > (1 + k)t$ for each t in this interval. Since $\alpha_0 > 0$, we only need $\alpha_0 > (1 + k)t_0$.

By Lemma 12, if $k \leq 1$, then η represents a connected symplectic submanifold (this imposes no condition); if $k > 1$, then if $\alpha_0 - 2kt_0 > 0$, η represents a connected symplectic submanifold.

For the next interval, let's start with $t = 0$. So when $0 \leq t < t_1$ (the reduced space M_{red} at the index 4 critical level is $\mathbb{C}P^2$.),

$$[\omega_t] = [\omega_{t_0}] - xt = (\alpha_0 - kt_0 - t)x + t_0 y.$$

To make $[\omega_t]$ contain a symplectic form for each t, we choose $\alpha_0 - (k+1)t_0 > t$ for $0 \leq t < t_1$.

After ϕ crosses the index 4 point, M_{red} is $\mathbb{C}P^2$ and $x + y$ is a generator of $H^2(\mathbb{C}P^2)$. If we do the computation starting from the top, we see that each cohomology class in this interval contains a symplectic form. To make the data compatible, we need

$$(\alpha_0 - kt_0 - t_1)x + t_0 y = t_2(x + y),$$

i.e., $t_2 = t_0$, and $\alpha_0 = t_1 + (k+1)t_0$.

In summary, we choose the parameters such that

$$\alpha_0 > 2kt_0, \quad \alpha_0 = t_1 + (k+1)t_0, \quad t_1 > 0 \quad (t_0 = t_2).$$

For each fixed k, we can adjust α_0, t_0, t_1 such that the above condition is satisfied This proves the existence of manifold(s) of type (3).

We have shown in [L] that manifold of type (4) is diffeomorphic to $\mathbb{C}P^3$.

For manifold of type (5), let us prove the existence for the case when ϕ crosses the index 4 point before it crosses the index 2 point.
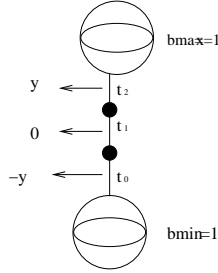
If $0 < t < t_0$, $M_{red} = \widetilde{\mathbb{C}P^2}$, then $[\omega_t] = \alpha_0 x + ty$, by Lemma 9, we may choose $\alpha_0 > t$ for each t in this interval to make $[\omega_t]$ contain symplectic forms. Since M_{red} at the index 4 point critical level is $\mathbb{C}P^2$ ($x + y \in H^2(\mathbb{C}P^2)$ is a generator), to make the cohomology class contain a symplectic form for $t = t_0$, by Lemma 7, we choose $\alpha_0 = t_0$.

Notice that the Euler class of the principal S^1 bundle over M_{red} for the t_1 interval is 0. We can do a similar computation starting from the top and choose $\tilde{\alpha} = t_2$, where $\tilde{\alpha}$ is the size of the maximum. To make the data compatible, (the cohomology class of the reduced symplectic form at the index 4 critical level and at the index 2 critical level should be the same) we should choose $t_0 = t_2$.

In summary, we can choose parameters such that

$$\alpha_0 = \tilde{\alpha} = t_0 = t_2.$$

This proves the existence of manifold of type (5).

For manifolds of type (6a), we will prove the existence for the case when $b_{min} = n = 2k$ is even (the reduced space is the trivial S^2 bundle over Σ_g at any non-extrem level.). The proof for the case when b_{min} is odd is similar (the reduced space is the non-trivial S^2 bundle over Σ_g at any non-extrem level).
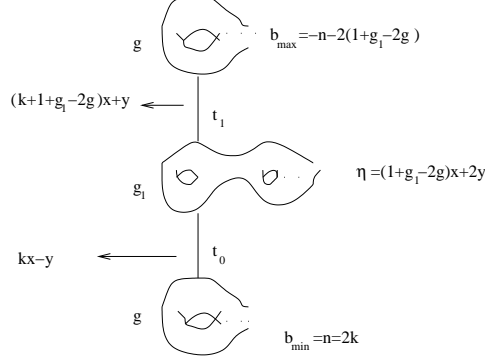
For $0 < t \le t_0$, $[\omega_t] = \alpha_0 x - (kx - y)t = (\alpha_0 - kt)x + ty$. By Lemma 9, we need $\alpha_0 - kt > 0$ for each t to make each cohomology class contain a symplectic form. Since $\alpha_0 > 0$, we only need $\alpha_0 > kt_0$.

By Lemma 11, when $1 + g_1 - 2g > 0$, the index 2 surface is a symplectic submanifold of the reduced space.

For $0 < t \le t_1$, $[\omega_t] = (\alpha_0 - kt_0)x + t_0 y - [(k+1+g_1-2g)x+y]t = [\alpha_0 - kt_0 - (k+1+g_1-2g)t]x + (t_0 - t)y$. We need $t_0 = t_1$, and $\alpha_0 - kt_0 - (k+1+g_1-2g)t_1 > 0$, i.e., $\alpha_0 - nt_0 > (1+g_1-2g)t_0$. If $k \ge 0$, it is easy to see this choice makes each cohomology class in this interval contain a symplectic form. If $k < 0$, then we need $\alpha_0 - kt_0 - kt > (1+g_1-2g)t$ for each $0 < t < t_0$ to make the cohomology classes contain symplectic forms.
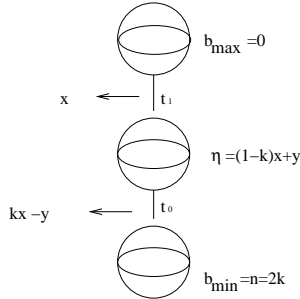
In summary, we make the following choices of the parameters:

$$\alpha_0 - kt_0 - kt > (1 + g_1 - 2g)t > 0, t \in [0, t_0].$$

We can adjust α_0 and t_0 such that the above condition is satisfied for any fixed k and g, g_1 such that $1 + g_1 - 2g > 0$. Therefore manifolds of type (6a) exist when $1 + g_1 - 2g > 0$.

For manifolds (6b) (twisted case), M_{red} is $S^2 \times S^2$.

When $b_{min} = n = 2k$ and $b_{max} = 0$, we have the following picture:

Notice that $t_0 > 0$, and $t_1 > 0$.

When $0 < t \leq t_0$, $[\omega_t] = \alpha_0 x - (kx - y)t = (\alpha_0 - kt)x + ty$, it is easy to see that when $\alpha_0 > kt_0$, each $[\omega_t]$ contains a symplectic form.

By Lemma 11, if $k \leq 1$, then η can be represented by a symplectic submanifold; if $k > 1$, then when $\alpha_0 - (n-1)t_0 > 0$, η can be represented by a symplectic submanifold.

When $0 \leq t \leq t_1$ (new start at $t = 0$), $[\omega_t] = (\alpha_0 - kt_0)x + t_0 y - xt = (\alpha_0 - kt_0 - t)x + t_0 y$. Since there is a twist, when $t = t_1$, we should have

$$\alpha_0 - kt_0 - t_1 = 0.$$

Notice that this kind of choice makes $[\omega_t]$ contain a symplectic form for each $0 < t < t_1$.

In summary, we can choose

$$\alpha_0 > (n-1)t_0, \quad t_1 = \alpha_0 - kt_0 > 0.$$

For each fixed k, the choice is achievable. This proves the existence of manifold of type (6b).

If this manifold exists, then its upside down also exists.

$\square$

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